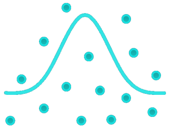
Hamburg University of Technology
Institute for Fluid Dynamics and Ship theory

Consistent kernel-based approximations and variable resolution schemes in SPH model

Marzia Leonardi, Jose M. Dominguez, Thomas Rung



Ephyslab - Universidade de Vigo (Ourense)

February 2^{nd}, 2018

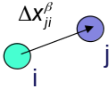
Outline

Outline

Consistency is a key issue

Convergence requires consistency (and not only ... Lax-Richtmyer)

- Order of consistency depends on **degree of polynomial** that can be reproduced by the kernel approximation.

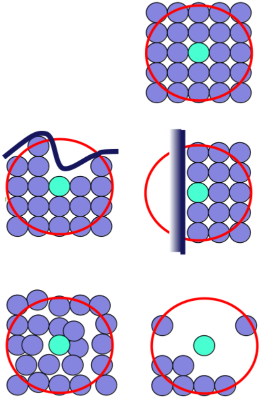


0^{th} order: $f(x) = a$	$\sum V_j W_{ij} = 1$	$\sum V_j W'^{\alpha}_{ij} = 0$
1^{st} order: $f(x) = ax + b$	$\sum V_j W_{ij} \Delta x_{ji}^{\alpha} = 0$	$\sum V_j W'^{\alpha}_{ij} \Delta x_{ji}^{\beta} = \delta^{\alpha\beta}$
2^{nd} order: $f(x) = ax^2 + bx + c$	$\sum V_j W_{ij} \Delta x_{ji}^{\alpha} \Delta x_{ji}^{\beta} = 0$	$\sum V_j W'^{\gamma}_{ij} \Delta x_{ji}^{\alpha} \Delta x_{ji}^{\beta} = 0$
...		

- **Different (moment) criteria** evolve depending on the **operator** that is observed (function, gradients, second derivatives, etc.)

Consistency

Consistency problems are associated to level of anti-symmetry of the **discrete** kernel sampling.



Practical relevance

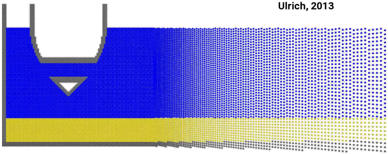
- Truncated kernel supports (e.g. boundaries or free surfaces);
- Isolated particles;
- Irregular particle distributions.

Life can be hard!

Problems related to many famous SPH topics with many famous work-arounds and models.

Variable Resolution

Many marine/coastal engineering problems (e.g. a ship hull in a full-scale water basin) require **a lot of particles**.



Practical relevance

- Large computational domains;
- Domains in which a confined region requires higher resolution;
- No high-perfomance computing avalaible.

Outline

Consistency

Manipulate/regularise discretisation/particle samples.
shift particles, introduce ghost/fixed wall particles, ...

Manipulate kernel-based approximation.

- Apply **Shepard** normalisation (0^{th}-order consistency) of kernels. Limited impact since function approximations are rare in PDEs.;
- Correct kernel + gradients following a minimization procedure (**MLS**);
- Correct kernel + gradients + higher derivatives from **Max.Entropy** option (Ortiz and Sukumar);
- Correct kernel + gradients + higher derivatives with **weitghed residuals** (Liu^2, Chen and Beraun);
- ...

Consistency

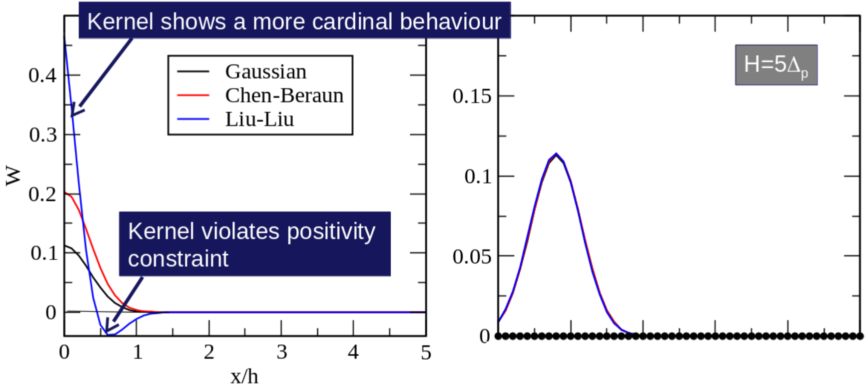
Manipulate/regularise discretisation/particle samples.
shift particles, introduce ghost/fixed wall particles, ...

Manipulate kernel-based approximation.
with MLS, MaxEnt, weighted residuals...
Disadvantages

- limited success for FS flows, due to violation of positivity constraint;
- conflict between conservation on particle level and consistency;
- significant computational effort to satisfy constraints
  (implicit $4X4$ [$10X10$] systems at each point and time for order 1[2]).

Consistency

Apparent kernels (1^{st} order)

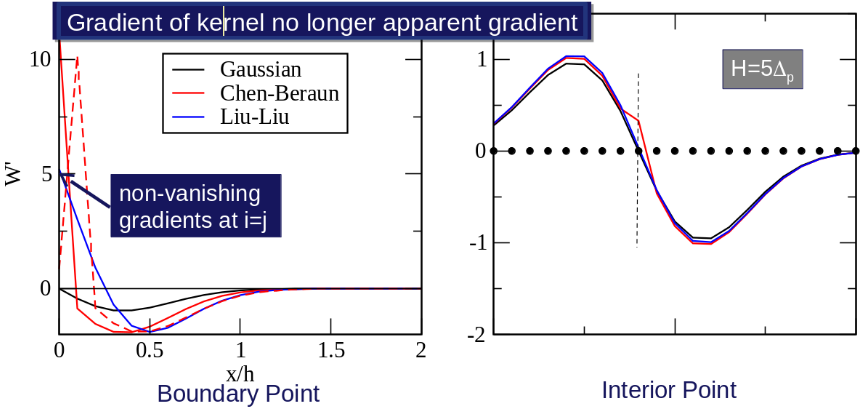


Boundary Point

Interior Point

Consistency

Apparent kernel gradients (1^{st} order)



Variable Resolution

Inhomogeneous particle distribution in the domain (e.g. Omidvar et al.)
NOT DYNAMIC: Retaining of particles properties/number during simulation time. Not suitable for violent dynamics and big deformations.

Merging/Splitting variable resolution techniques (e.g. Barcarolo, Feldman, Vacondio et al.)
DYNAMIC: Particles number, properties and locations change in time (in compliance with conservation of mass and momentum).

Wishful thinking

We desire

- Cheap, preferably explicit correction;
- Focus on gradient approximation;
- Directly applied in discrete space (inherently adjust to actual sampling);
- Easy and natural access to Dirichlet and Neumann conditions;
- Build from familiar kernels and Re-castable into familiar frame (apparent W_{ij} and W'_{ij}).

We also desire

- Coupling of consistent SPH approximations with a variable resolution scheme;
- A dynamic (or partially dynamic) variable resolution approach.

Outline

Explicit Consistency Correction

Desired achievable order

- 1^{st} order sufficient for most cases;
- 2^{nd} order has debatable features (jumps and inevitable loss of positivity).

Focal point

- is on the gradient rather than functions (no need); build from standard kernels

Design principle

- **3 conditions** can be implemented explicitly in discrete space (+3 loops);
- this is sufficient **to perform 3 x 1D up to** 2_{nd} **order** but not 1_{st} order 3D;
- correction refer to kernel counterparts (symmetrical - unsymmetrical);
- priority given to robustness (**mollified correction**) rather than accuracy.

Explicit Consistency Correction

Kernel Gradient Correction (example)
Gradient is anti-symmetric - thus correction is symmetric (= kernel)
(Truncation of W' strives for symmetry - truncation of W strives for opposite)
As opposed to Chen and Beraun the correction is mollified around the focal point

$$\tilde{W}'_{ij} = W'_{ij} + \alpha W_{ij}$$

Conditions (3 X 1D, i.e. ignore symmetry conditions normal to differentiation)

$$\sum_j V_j \tilde{W}'_{ij} = 0 \qquad \sum_j V_j \tilde{W}'_{ij}(x_j - x_i) = 1 \qquad \sum_j V_j \tilde{W}'_{ij}(x_j - x_i)^2 = 0$$

Normalisation conditions are always implemented at the end.

Explicit Consistency Correction

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Start with 1_{st} order Consistency

$$\beta = \sum_j V_j W'_{ij} \qquad \gamma = \sum_j V_j W_{ij} \qquad \rightarrow \qquad \hat{W}'_{ij} = W'_{ij} - \frac{\beta}{\gamma} W_{ij}$$

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Perform final normalisation which does not alter previous constraints

$$\tilde{W}'_{ij} = \frac{\hat{W}'_{ij}}{\sum_j V_j \hat{W}'_{ij}(x_j - x_i)}$$

Explicit Consistency Correction

Kernel Gradient Correction (example)
Gradient is anti-symmetric - thus correction is symmetric (= kernel)
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Conditions (3 X 1D)

$$\sum_j V_j \tilde{W}'_{ij} = 0 \qquad \sum_j V_j \tilde{W}'_{ij}(x_j - x_i) = 1 \qquad \boxed{\sum_j V_j \tilde{W}'_{ij}(x_j - x_i)^2 = 0}$$

Also 2_{nd} Order Consistency can be derived in a similar way.

$$\zeta = \sum_j V_j W_{ij}(x_j - x_i)^2 \qquad \psi = \sum_j V_j W'_{ij}(x_j - x_i)^2 \qquad \rightarrow \qquad \hat{W}'_{ij} = W'_{ij} - \frac{\psi}{\zeta} W_{ij}$$

Explicit Consistency Correction

Kernel Gradient Correction (example)
Gradient is anti-symmetric - thus correction is symmetric (= kernel)
(Truncation of W' strives for symmetry - truncation of W strives for opposite)
As opposed to Chen and Beraun the correction is mollified around the focal point

$$\tilde{W}'_{ij} = W'_{ij} + \alpha W_{ij}$$

Conditions (3 X 1D)

$$\sum_j V_j \tilde{W}'_{ij} = 0 \qquad \boxed{\sum_j V_j \tilde{W}'_{ij}(x_j - x_i) = 1} \qquad \sum_j V_j \tilde{W}'_{ij}(x_j - x_i)^2 = 0$$

Perform final normalisation which does not alter previous constraints

$$\tilde{W}'_{ij} = \frac{\hat{W}'_{ij}}{\sum_j V_j \hat{W}'_{ij}(x_j - x_i)}$$

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Kernel Gradient Correction (example)
Gradient is anti-symmetric - thus correction is symmetric (= kernel)
(Truncation of W' strives for symmetry - truncation of W strives for opposite)
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$$\tilde{W}'_{ij} = W'_{ij} + \alpha W_{ij}$$

Conditions (3 X 1D)

$$\boxed{\sum_j V_j \tilde{W}'_{ij} = 0} \qquad \sum_j V_j \tilde{W}'_{ij}(x_j - x_i) = 1 \qquad \sum_j V_j \tilde{W}'_{ij}(x_j - x_i)^2 = 0$$

0_{th}-**order condition** implemented at focal point W'_{ii} since this has **no influence** on second order constraint (unmollified correction)

$$\tilde{W}'_{ii} = -\sum_{j \neq i} \tilde{W}'_{ij}$$

Explicit Consistency Correction - 1^{st} order

Kernel Function

$$\tilde{W}_{ij} = W_{ij} + \alpha W_{ij}^{'}$$

$$\sum_j V_j \hat{W}_{ij}(x_j - x_i) = 0 \rightarrow \alpha = -\frac{\sum_j V_j W_{ij}(x_j - x_i)}{\sum_j V_j W_{ij}^{'}(x_j - x_i)}$$

$$\sum_j V_j \tilde{W}_{ij} = 1 \rightarrow \tilde{W}_{ij} = \frac{\hat{W}_{ij}}{\sum_j V_j \hat{W}_{ij}}$$

Kernel Gradient

$$\tilde{W}_{ij}^{'} = W_{ij}^{'} + \alpha W_{ij}$$

$$\sum_j V_j \hat{W}_{ij}^{'} = 0 \rightarrow \alpha = -\frac{\sum_j V_j W_{ij}^{'}}{\sum_j V_j W_{ij}}$$

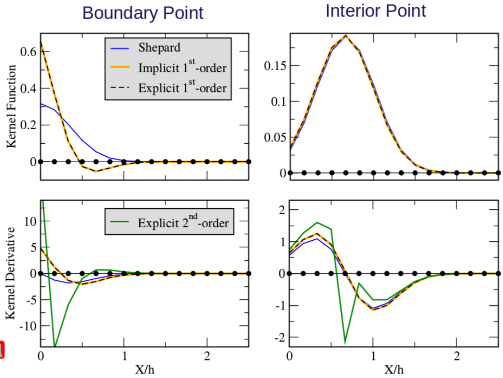
$$\sum_j V_j \tilde{W}_{ij}^{'}(x_j - x_i) = 1 \rightarrow \tilde{W}_{ij}^{'} = \frac{\hat{W}_{ij}^{'}}{\sum_j V_j \hat{W}_{ij}^{'}(x_j - x_i)}$$

Key idea: to bias a (sufficient) quantity α of an **even function** in case of an **uneven function**, and vice versa.

The consistency constraints (0^{th}, 1^{st} and 2^{nd}-order conditions) give a measure of the needed α.

Explicit Consistency Correction

Comparison of different approaches



Explicit Consistency Correction

Some remarks

- Kernel,kernel gradient and higher-order correction are performed individually in the discrete space;
- there is no simple formal link between these properties anymore;
- Kernel function symmetry / kernel gradient anti-symmetry is lost. Conservation is also lost;
- Negative kernel values occur;
- **Neumann condition can be imposed by means of equivalent Dirichlet condition**

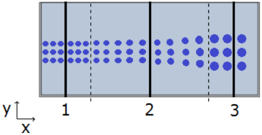
$$\overline{f}_i' = \sum_{j \neq i} V_j \tilde{W}_{ij}' f_j + V_i \tilde{W}_{ii}' f_i \qquad \rightarrow \qquad \boxed{f_i = \frac{\overline{f_i'} - \sum_{j \neq i} V_j \tilde{W}_{ij}' f_j}{V_i \tilde{W}_{ii}'}}$$

Outline

Variable Resolution

Eulerian Variable Resolution Scheme

Following a route by Ulrich et al.



The analytical definition of the kernel function (and of its gradient) is supplemented by a term addressing the **spatial variation of the kernel**

$$W_{ij}(r, \boxed{h}) \qquad \nabla W_{ij} = \frac{\partial W_{ij}}{\partial r} \nabla r \boxed{+ \frac{\partial W_{ij}}{\partial h} \nabla h}$$

The Navier-Stokes equation are also supplemented with appropriate source terms, accounting for the **variation of particle masses**

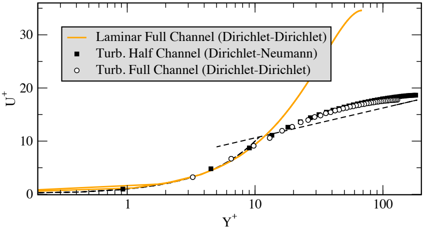
$$S_{CE} = \sum_{j=1}^{N} V_j v_j^{\alpha} \frac{\partial \rho_j}{\partial x_j^{\alpha}} W_{ij} = \sum_{j=1}^{N} v_j^{\alpha} \frac{\partial m}{\partial x_j^{\alpha}} W_{ij} \qquad S_{ME} = -\sum_{j=1}^{N} \frac{v_i^{\alpha}}{m_i} \frac{m_j}{\rho_j} \left(v_j^{\beta} \frac{\partial m}{\partial x_j^{\beta}} W_{ij} \right)$$

Outline

Validation

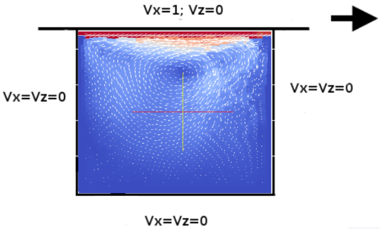
Channel flow (laminar and turbulent $Re_\tau = 170$)

Different condition on the upper side of the turbulent channel flow (Dirichlet vs. Neumann; different inhomogeneous particle distributions)



Verification

Lid Driven Cavity Flow (2D, with 3 different Re numbers)

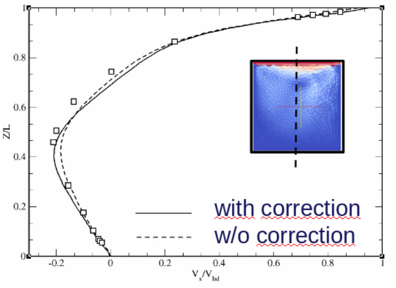


Simulation Parameters				
Test Case	ΔP[m]	Re	L [m]	Vx[m/s]
A	0.01- 0.007 -0.005	100	1	1
B	0.01- 0.007 -0.005	400	1	1
C	0.007- 0.005	1000	1	1

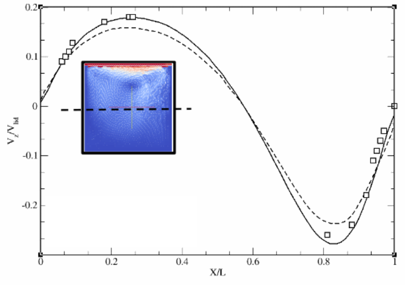
Verification

Lid Driven Cavity Flow at $Re = 100$

Comparison of vertical and horizontal profiles at t = 54 s



vertical

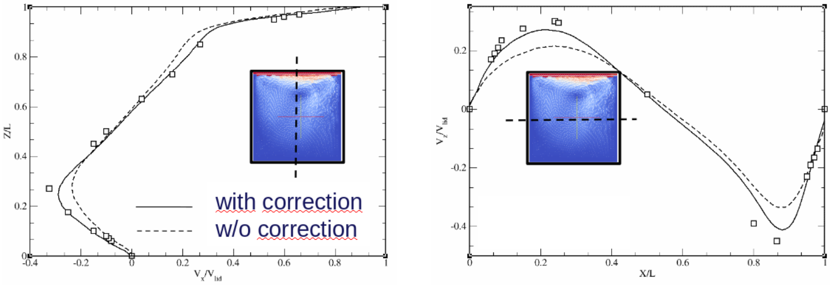


horizontal

Verification

Lid Driven Cavity Flow at $Re = 400$

Comparison of vertical and horizontal profiles at t = 54 s

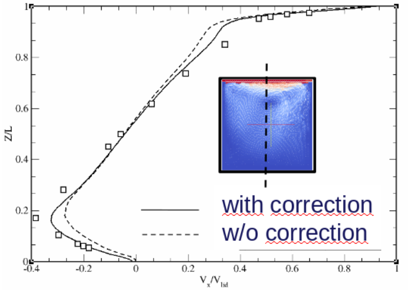


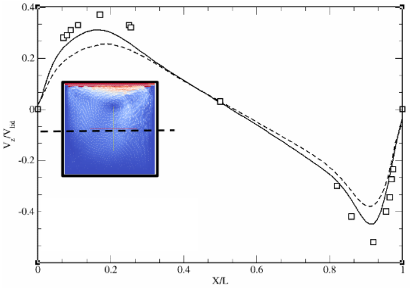
vertical horizontal

Verification

Lid Driven Cavity Flow at $Re = 1000$

Comparison of vertical and horizontal profiles at $t = 54$ s



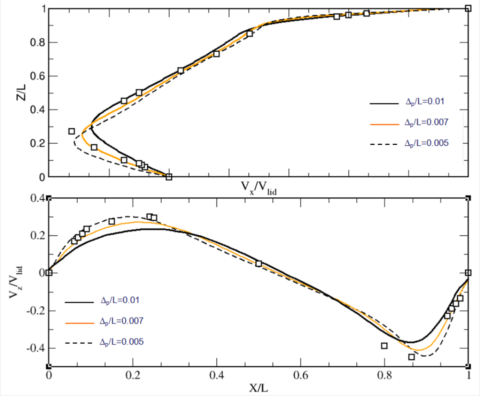


vertical

horizontal

Verification

Convergence study for $Re = 400$ (with gradient correction)



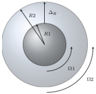
Validation

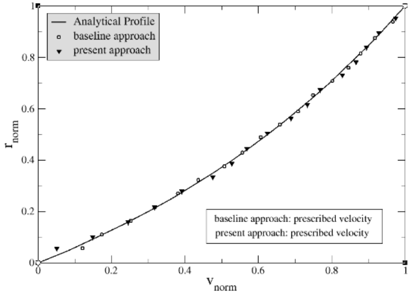
Axisymmetric Couette flow (with and without application of Neumann condition)

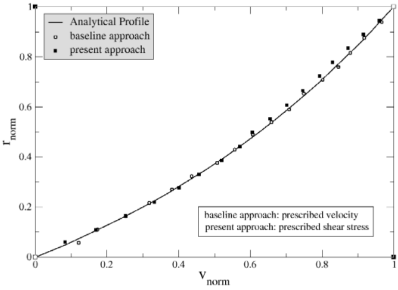
$R_1 = 1m$; $R_2 = 2m$

$\omega_1 = 0.005$; $\omega_2 = 0.001$

$dp = 0.05m$



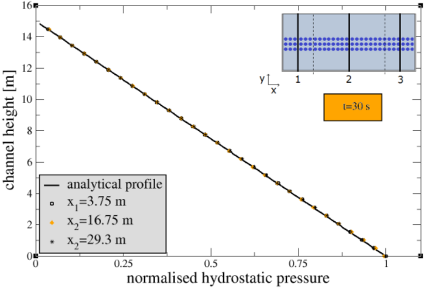


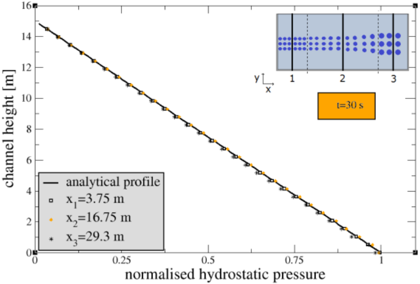


Validation -coupling-

Hydrostatic case (water basin)

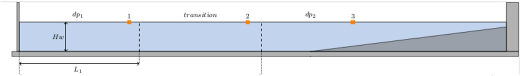
L= 30 m; H = 15 m
profiles at 3 sections (x_1,x_2 and x_3)
dp=0.25 m; 8200 particles circa

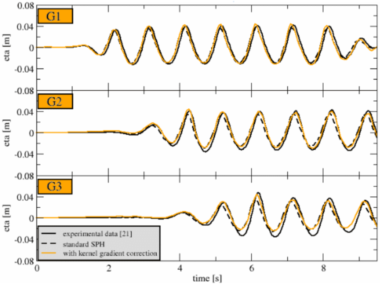


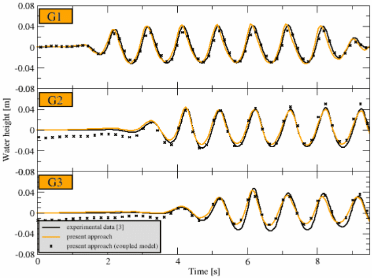


Verification -coupling-

Wavemaker test case (comparison with experimental data)







Outline

Remarks

- **Significant response to kernel gradient correction** on preliminary validation 1D and 2D test cases. $\rightarrow$ The consistency of the kernel gradient has been increased at a **low computational cost** (max 3 loops);
- It is possible to apply **Neumann boundary condition** $\rightarrow$ gradient of the kernel function is not 0 in the focal point!
- **No additional row of particles** have been used to mimic the wall, as remedy to boundary truncation;
- The coupling of KGC with VR delivered **good results** with both hydrostatic and dynamic cases;
- The use of VR **reduces significantly** the total amount of particles needed (circa 50 percent).

Future Developments

- It will be will be tested using **3D test cases/more complex geometries**;
- Improvement of **Pre**- and **Postprocessing procedures** (slightly different from standard DualSHysics tools, due to variable resolution);
- **Upgrade** to the last version of DualSPHysics is planned.

Thank you
for your kind attention!

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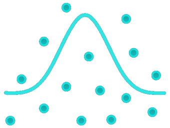

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