A brief description of the particle finite element method (PFEM2). Extensions to free surface flows.

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Motivations for using PFEM.

- Combines convective particle movement and fixed mesh resolution.
- Accurate description of free surface flows.

Two novel features:

- Larger time steps compared to other similar numerical tools.
- Improved versions of discontinuous and continuous enriched basis functions for the pressure field.

Objectives

- Shorter computational times while the solution accuracy is maintained.
- Free surface should be reconstructed without artificial diffusion or undesired numerical effects.
- Validation of free surface flows.
  - 2D and 3D cases.
  - Low and high Reynolds numbers.

Compared to well validated numerical alternatives and experimental measurements.

From PFEM (Idelsohn-2004) to PFEM2

- Robust method where Lagrangian particles and meshing processes are alternated.
- FEM structure supports the differential equation solvers.
- Non-material points transport fixed intensive properties of the fluid.

Current *Fixed Mesh*-PFEM-2:

- X-IVAS (eXplicit Integration following the Velocity and Acceleration Streamlines).
- The initial background mesh is preserved, avoiding remeshing at each time-step.
- Mesh nodes and moving particles interchange information through interpolation algorithms.

PFEM Algorithm.

$$\begin{aligned} \nabla \cdot \mathbf{v} &= 0 \\ \rho \frac{D\mathbf{v}}{Dt} &= -\nabla p + \mu \nabla^2 \mathbf{v} + \mathbf{f} \end{aligned}$$

Velocity and pressure are decoupled by a **fractional step method**.

1. Predictor.
   1. Acceleration calculation at t^n $\mathbf{a}_j^n$ on the mesh nodes.
   2. The X-IVAS stage to convect the fluid properties using the particles.
   3. The projection of the particle data to the mesh nodes.
   4. The implicit calculation of the diffusion term.
2. Poisson equation.
3. Corrector.

Predictor: Acceleration step.

It is assumed that all fluid variables are known at time t_n for both the particles and the mesh nodes.

$$\int_\Omega \mathbf{a}_\tau^n \psi_j \, d\Omega = \int_\Omega \mu \nabla^2 \mathbf{v}^n \psi_j \, d\Omega = -\int_\Omega \mu \nabla \mathbf{v}^n \nabla \psi_j \, d\Omega + \int_\Gamma \mu \nabla \mathbf{v}^n \psi_j \cdot \mathbf{n} \, d\Gamma$$

$$\int_\Omega \mathbf{a}_p^n \psi_j \; d\Omega = -\int_\Omega \nabla p^n \psi_j \; d\Omega$$

$$\mathbf{a}^n = \mathbf{a}_p^n + (1-\theta)\mathbf{a}_\tau^n$$

- Where θ is a numerical parameter that rules the explicitness of the viscous term in the algorithm.
- Lamping the mass matrix is often used for both problems.

Predictor: X-IVAS stage.

Evaluates the new particle position $\mathbf{x}_p^{n+1}$ and intermediate velocity $\widehat{\mathbf{v}}_p^{n+1}$ following the velocity streamlines at t^n

$$\mathbf{x}_p^{n+1} = \mathbf{x}_p^n + \int_n^{n+1} \mathbf{v}^n(\mathbf{x}_p^t)\; dt$$

$$\widehat{\mathbf{v}}_p^{n+1} = \mathbf{v}_p^n + \int_n^{n+1} \left[\mathbf{a}^n(\mathbf{x}_p^t) + \mathbf{f}^t(\mathbf{x}_p^t)\right]\; dt$$

N sub-stepping integration, where $N\delta t = \Delta t$

$$\mathbf{x}_p^{n+1} = \mathbf{x}_p^n + \sum_{i=1}^{N} \mathbf{v}^n(\mathbf{x}_p^{n+\frac{i}{N}})\delta t$$

$$\widehat{\mathbf{v}}_p^{n+1} = \mathbf{v}_p^n + \sum_{i=1}^{N} \left[\mathbf{a}^n(\mathbf{x}_p^{n+\frac{i}{N}}) + \mathbf{f}^n(\mathbf{x}_p^{n+\frac{i}{N}})\right]\delta t$$

Predictor: Projection stage.

Projects velocity from the particles onto the mesh nodes:

$$\widehat{\mathbf{v}}_j^{n+1} = \frac{\sum_p \widehat{\mathbf{v}}_p^{n+1} W(\mathbf{x}_j - \mathbf{x}_p^{n+1})}{\sum_p W(\mathbf{x}_j - \mathbf{x}_p^{n+1})}$$

Where the Wendland kernel function was used for the projections.

Predictor: Implicit Viscosity Stage.

Implicit correction of the viscous diffusion. The fractional velocity $\widehat{\mathbf{v}}_j^{n+1}$ is found on the mesh nodes.

$$\int_\Omega \widehat{\mathbf{v}}_j^{n+1}\psi_j \; d\Omega = \int_\Omega \widehat{\widehat{\mathbf{v}}}_j^{n+1}\psi_j \; d\Omega + \theta\Delta t \int_\Omega \mu\nabla^2\widehat{\mathbf{v}}_j^{n+1}\psi_j \; d\Omega$$

Poisson Stage.

Computes the pressure correction δp^{n+1} on the mesh nodes by solving the Poisson equation.

$$
\begin{aligned}
\int_\Omega \nabla \cdot \left[\frac{\Delta t}{\rho} \nabla(\delta p^{n+1}) \right] \phi_j \; d\Omega &= \int_\Omega \nabla \cdot \widehat{\mathbf{v}}_j^{n+1} \phi_j \; d\Omega \\
\frac{\partial \delta p^{n+1}}{\partial n} &= 0 \quad in \quad \Gamma_D \\
\delta p^{n+1} &= 0 \quad in \quad \Gamma_N
\end{aligned}
$$

Pressure at time t_{n+1} is updated as $p^{n+1} = p^n + \delta p^{n+1}$.

Correction Stage.

Updates the mesh and particle velocity with the pressure and diffusion corrections:

$$\int_\Omega \rho_j \mathbf{v}_j^{n+1} \psi_j \ d\Omega \ = \ \int_\Omega \rho_j \widehat{\mathbf{v}}_j^{n+1} \psi_j \ d\Omega \ - \Delta t \int_\Omega \nabla \delta p^{n+1} \psi_j \ d\Omega$$

$$\rho_p \mathbf{v}_p^{n+1} \ = \ \rho_p \widehat{\widehat{\mathbf{v}}}_p^{n+1} + \sum_j \delta \mathbf{v}_j^{n+1} \psi_j(\mathbf{x}_p^{n+1})$$

where $\delta \mathbf{v}_j^{n+1} = \mathbf{v}_j^{n+1} - \widehat{\widehat{\mathbf{v}}}_j^{n+1}$.

Summary: PFEM2 Algorithm.

Nodes	Acceleration $\rightarrow$ $\mathbf{a}_\tau + \mathbf{a}_p$
Particles	X-IVAS: $\mathbf{x}^{n+1}$, $\widehat{\widehat{\mathbf{v}}}^{n+1}$
Particles $\rightarrow$ Nodes	Projection $\widehat{\widehat{\mathbf{v}}}_j^{n+1}$
Nodes	Implicit Viscosity:$\widehat{\mathbf{v}}_j^{n+1}$
Nodes	Poisson:p_j^{n+1}
Nodes	Correction
Nodes $\rightarrow$ Particles	Interpolation:$\mathbf{v}_j^{n+1}$

Free surface flows.

- Two different fluids separated by an interface are considered
- It is essential that the interface remains sharp.
- Large jumps of fluid density and viscosity across the interface.
- Each particle p carries the information of the fluid to which it was initially assigned λ_p.

$$\lambda = \begin{cases} -1 \text{ Fluid A} \\ 1 \text{ Fluid B} \end{cases}$$

- This value is advected $\frac{D\lambda}{Dt} = 0$, i.e. each particle keeps its λ value.
- λ is projected to the mesh nodes. $\lambda_j \neq \lambda_p = \pm 1$ that the particles transport.
- Free-surface interface $\lambda = 0$

Free surface flows: velocity of the moving particles.

Two situations:

1. All the nodes of the hosting element have the same density as the fluid particle $\Rightarrow$ typical finite element interpolation.
2. One or more nodes have a different density than the fluid particle(close to the interface). If $\rho_1/\rho_2 > \alpha$, two situations can appear:
   - $\rho_p = \rho_2$ (light particle tracking). The velocity used in the particle movement will be interpolated.
   - $\rho_p = \rho_1$ (heavy particle tracking). Depending on the value of

$$A = \sum_{i \in (\rho_i = \rho_p)} \psi_i$$

     where the sum is limited to the hosting nodes that have the same density as the particle, we can have 2 possibilities:

Free surface flows.

- $A < \beta$ the gravity force will be included in the computation of the particle trajectory, which will finally be computed as a parabolic motion.
- $A > \beta$, the interpolation is restricted to the hosting nodes i that have the same density as the particle $\rho_i = \rho_p$.

Free surface flows. Enriched basis functions.

- We are trying to reproduce a continuous function such as *pressure* but with discountinous derivative.
- Normally FEM basis functions are continuous, consequently a **enriched** space must be used.

Conclusion.

Life is too short, other conclusions are just dummy ideas.